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A Characterization of PSL$(2, q)$, $q = 5, 7$

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Abstract. In this short note we prove that the finite non-abelian simple groups $\mathrm{PSL}(2, q)$, where $q = 5, 7$, are determined by their posets of classes of isomorphic subgroups. Several interesting open problems are also formulated.

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1 Introduction

In group theory there are many ways to recognize the finite simple groups: by spectrum, by prime graph, by non-commuting graph, by subgroup lattices, and so on. Another way to recognize the first two non-abelian simple groups, $\mathrm{PSL}(2, q)$ with $q = 5, 7$, is presented in the following. It uses the poset $\mathrm{Iso}(G)$ of classes of isomorphic subgroups of a group G (see [4]):

$$\mathrm{Iso}(G) = \{[H] \mid H \leq G\}, \quad \text{where } [H] = \{K \leq G \mid K \cong H\}.$$

Recall that $\mathrm{Iso}(G)$ is partially ordered by

$$[H_1] \leq [H_2] \Longleftrightarrow K_1 \subseteq K_2 \text{ for some } K_1 \in [H_1] \text{ and } K_2 \in [H_2].$$

Obviously, all finite abelian simple groups G have the same poset $\mathrm{Iso}(G)$ (a chain of length 1) and consequently they cannot be recognized in this way. In the non-abelian case the situation is better, as our main result shows.

Theorem. *Let $G_0 \in \{\mathrm{PSL}(2, 5), \mathrm{PSL}(2, 7)\}$ and G be a finite group such that $\mathrm{Iso}(G) \cong \mathrm{Iso}(G_0)$. Then $G \cong G_0$.*

Moreover, we have verified by computer that any group G of order less than 60 is not determined by $\mathrm{Iso}(G)$. In other words, $\mathrm{PSL}(2, 5)$ is the first finite group that can be recognized by its poset of classes of isomorphic subgroups. Inspired by these results, we came up with the following conjecture.

Conjecture. For every non-trivial finite solvable group G_0 there are infinitely many finite groups G such that $G \ncong G_0$ and $\mathrm{Iso}(G) \cong \mathrm{Iso}(G_0)$.

The following question is also natural.

Question. Are all finite non-abelian simple groups determined by their posets of classes of isomorphic subgroups?

2 Proof of the Main Result

We start with the following easy but important lemma.

Lemma. *Let G_0 and G be two finite groups, $f : \mathrm{Iso}(G_0) \to \mathrm{Iso}(G)$ be a poset isomorphism, and H_0 and H be two subgroups of G_0 and G, respectively, such that $f([H_0]) = [H]$. Then the following statements hold:*

(a) $\mathrm{Iso}(H_0) \cong \mathrm{Iso}(H)$.

(b) *If $|H_0| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i, $i = 1, 2, \ldots, k$, are distinct primes, then*

$$|H| = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k}$$

for some distinct primes $q_1, q_2, \ldots, q_k$.

(c) *If all isomorphic copies of H_0 are maximal subgroups of G_0, then H is a maximal subgroup of G.*

Proof. (a) It is obvious that f induces a poset isomorphism from $\mathrm{Iso}(H_0)$ to $\mathrm{Iso}(H)$.

(b) This follows from (a) and [4, Theorem 3.2].

(c) Let K be a subgroup of G such that $H \subset K \subset G$. Then $[H] < [K] < [G]$, and therefore

$$f^{-1}([H]) < f^{-1}([K]) < f^{-1}([G]),$$

i.e., $[H_0] < [K_0] < [G_0]$, where $[K_0] = f^{-1}([K])$. It follows that there are $H_0' \cong H_0$ and $K_0' \cong K_0$ such that $H_0' \subset K_0' \subset G_0$, a contradiction. □

Remark. The assumption in (c) of the above lemma is justified, because a subgroup H_0' isomorphic to a maximal subgroup H_0 of a group G_0 is not necessarily maximal. For example, let G_0 be a finite non-abelian simple group,

$$H_0 = \{(x, x) \mid x \in G_0\} \quad \text{and} \quad H_0' = G_0 \times 1.$$

Then H_0 is maximal in G_0, $H_0' \cong H_0$ ($\cong G_0$), but clearly H_0' is not maximal.

We are now able to prove our main result.

Proof of Theorem. Assume first that $G_0 = \mathrm{PSL}(2, 5)$. By the above lemma we have $|G| = p^2 qr$, where p, q, r are distinct primes. It is well known that the maximal subgroups of $\mathrm{PSL}(2, 5)$ are of order 12 (isomorphic with A_4), 10 (isomorphic with

D_{10}), and 6 (isomorphic with S_3). Moreover, any subgroup of PSL$(2, 5)$ isomorphic with A_4, D_{10} or S_3 is also maximal. Therefore, if $f : \mathrm{Iso}(G_0) \to \mathrm{Iso}(G)$ is a poset isomorphism, and

$$[M_1] = f([A_4]), \quad [M_2] = f([D_{10}]) \quad \text{and} \quad [M_3] = f([S_3]),$$

then M_1, M_2 and M_3 are maximal subgroups of G of order p^2q (or p^2r), pq and pr, respectively. Now suppose that G contains a maximal subgroup M which is not isomorphic with M_1, M_2 or M_3. Then

$$[M] < [M_1], \quad [M] < [M_2] \quad \text{or} \quad [M] < [M_3]$$

because $[M_1]$, $[M_2]$ and $[M_3]$ are the maximal elements of $\mathrm{Iso}(G)$. This implies that $|M|$ is a proper divisor of $|M_1|$, $|M_2|$ or $|M_3|$, i.e., $|M| \in \{p, q, r, pq\,(\text{or}\,pr)\}$. Consequently, the orders of maximal subgroups of G are

$$p^2q \ \ (\text{or } p^2r), \quad pq, \quad pr,$$

and possibly proper divisors of these numbers.

Thus G has no Sylow system (it cannot have subgroups of order qr), and therefore it is not solvable. Since PSL$(2, 5) \cong A_5$ is the unique non-solvable group of order p^2qr (see e.g. [1]), it follows that $G \cong G_0$, as desired.

Assume next that $G_0 =$ PSL$(2, 7)$. Then $|G| = p^3qr$, where p, q, r are distinct primes. Since the maximal subgroups of PSL$(2, 5)$ are of order 24 (isomorphic with S_4) and 21 (isomorphic with the Frobenius group of order 21), a similar argument implies that the orders of maximal subgroups of G are

$$p^3q \ \ (\text{or } p^3r), \quad qr,$$

and possibly proper divisors of these numbers.

Again, G has no Sylow system (it cannot have simultaneously subgroups of order p^3q and p^3r), and thus it is not solvable. This shows that G has a composition factor, say G_1/G_2, which is a non-abelian simple group. Then $|G_1/G_2|$ is even. Moreover, Theorem 1.35 of [3] implies that it is divisible by 4. This leads to $p = 2$, i.e., $|G| = 8qr$. Consequently, by [2, pp. 12–14], we have either

$$G_1/G_2 \cong \mathrm{PSL}(2, 5) \quad \text{or} \quad G_1/G_2 \cong \mathrm{PSL}(2, 7).$$

In the first case we infer that G has order 120, and so it is isomorphic to one of the following groups: S_5, $A_5 \times \mathbb{Z}_2$, and SL$(2, 5)$. This is impossible because none of these groups has any maximal subgroup of order 15. Hence, $G_1/G_2 \cong$ PSL$(2, 7)$, which shows $G \cong G_0$. This completes the proof. $\square$

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